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# $\mu_n$ -actions on K3 surfaces in positive characteristic

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## Abstract: We study $\mu_p$ -actions on K3 surfaces in char $p$ .

NB: Smooth K3 surfaces admit no  $\mu_p$ -actions since they admit no global derivations. However RDP K3 surfaces may admit some.

### Preliminaries

- A *K3 surface* is a proper smooth (algebraic) surface  $X$  over a field with  $\Omega_X^2 \cong \mathcal{O}_X$  and  $H^0(X, \mathcal{O}_X) = 0$ .
- An *RDP K3 surface* is a proper surface  $X$  with only RDP (rational double point) singularities whose resolution  $\tilde{X}$  is a K3.
- Definition.** An automorphism of a K3  $X$  is **symplectic** if it acts on the (1-dim vector space)  $H^0(X, \Omega_X^2)$  trivially.

### Nikulin: actions in char 0

- $G$ : finite abelian group,  $X$ : K3.  
 $G \curvearrowright X$ : a *symplectic* action.  
 Then •  $\text{Fix}(G)$  is isolated.  
 •  $X/G$  is an RDP K3.  
 • If  $G = \mathbb{Z}/n\mathbb{Z}$  with  $n > 1$ , then  $n \leq 8$  and  

$$\# \text{Fix}(G) = \frac{24}{n} \prod_{l: \text{prime}, l|n} \frac{l}{l+1}$$

$$= 8, 6, 4, 4, 2, 3, 2 \quad (n = 2, 3, 4, 5, 6, 7, 8).$$

### Non-symplectic quotients

are either birational to Enriques, or rational.

### Actions in char $p$ ?

- Nikulin's results hold in char  $p > 0$  provided the order of  $G$  is prime to  $p$ .  
 However, **automorphisms of order  $p$  are automatically symplectic**, since there are no nontrivial  $p$ -th root of unity in char  $p$ .  
 •  $\exists$  order  $p$  auto with 1-dim fixed locus,  
 •  $\exists$  order  $p$  auto with non-K3 quotient.  
**Remark.**  $\exists$  order  $p$  auto only if  $p \leq 11$ .  
 For more discussions see Dolgachev–Keum.

### Keum: Orders of auto.

- $S_{\text{cyc}}(p) := \{n \mid \mathbb{Z}/n\mathbb{Z} \curvearrowright \exists X \text{ K3 in char } p\}$ .  
 Keum determined this set for  $p \neq 2, 3$ .  
 •  $S_{\text{cyc}}(0) = \{n \mid \phi(n) \leq 20\}$   
 $= \{1, \dots, 22, 24, 25, 26, 27, 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 48, 50, 54, 60, 66\}$ ,  
 •  $S_{\text{cyc}}(p) = S_{\text{cyc}}(0) \setminus E_p$  if  $p \geq 5$ , where  

$$E_p = \begin{cases} \{p, 2p\} & \text{if } p = 13, 17, 19, \\ \{44\} & \text{if } p = 11, \\ \{25, 50, 60\} & \text{if } p = 5, \\ \emptyset & \text{if } p = 7 \text{ or } p \geq 23. \end{cases}$$
  
 • Moreover the prime-to- $p$  elements of  $S_{\text{cyc}}(p)$  coincide with those of  $S_{\text{cyc}}(0)$ , for all  $p \geq 2$ .

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### Main Def: symplecticness

- Action  $\mu_n \curvearrowright \text{Spec } B$  (affine scheme)  
 $\longleftrightarrow$  a decomposition  $B = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} B_i$  of vector spaces satisfying  $B_i B_j \subset B_{i+j}$ .  
 $\longrightarrow$  a decomp.  $\Omega_{B/k}^* = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} (\Omega_{B/k}^*)^i$ .  
 We say that  $\text{wt}(b) = i$  if  $b \in B_i$ .  
 Similar for  $\mu_n$ -actions on schemes.  
**Remark.** If  $p \nmid n$ , then  $\mu_n$ -action is equivalent to the action of the cyclic group  $\mu_n(k)$ , and  $B_i$  is the eigenspace for the  $\mu_n(k)$ -action with eigenvalue  $i$ :  $\mu_n(k) \ni g \mapsto g^i \in k^*$ .  
**Definition.** We call an action  $\mu_n \curvearrowright X$  on an RDP K3 to be **symplectic** if the decomposition of (1-dim vector space)  $H^0(X^{\text{sm}}, \Omega_X^2)$  is concentrated on  $i = 0 \in \mathbb{Z}/n\mathbb{Z}$ .  
**Remark.**  $H^0(X^{\text{sm}}, \Omega_X^2) \cong H^0(\tilde{X}, \Omega_{\tilde{X}}^2)$  for an RDP K3  $X$  (hence 1-dim).  
**Remark.** Equivalent to the classical definition if  $p \nmid n$  (in which case  $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ ).

### Theorem A

- $X$ : RDP K3, with an action  $\mu_n \curvearrowright X$ .  
 • symplectic  $\implies X/\mu_n$ : RDP K3.  
 • non-symplectic  
 $\implies X/\mu_n$ : RDP Enriques or rational.  
 •  $n = p$ , non-symplectic, fixed-point-free  
 $\implies X/\mu_p$ : RDP Enriques  $\implies p = 2$ .  
 •  $n = p$ , non-symplectic, not fixed-point-free  
 $\implies X/\mu_p$ : rational surface.

### Theorem B

- $X$ : RDP K3.  
 $\mu_n \curvearrowright X$ : *symplectic* action,  $n > 1$ .  
 Then •  $n \leq 8$ , •  $\text{Fix}(G)$  is isolated,  
 •  $\# \text{Fix}(G)$  = same as in Nikulin's theorem, when counted with suitable multiplicity.

### Theorem C

- $S_\mu(p) := \{n \mid \mu_n \curvearrowright \exists X \text{ RDP K3 in char } p\}$ .  
 •  $S_\mu(0) = S_{\text{cyc}}(0) = \downarrow$ .  
 •  $S_\mu(p) = S_\mu(0) \setminus E'_p$ , where  

$$E'_p = \begin{cases} \{33, 66\} & \text{if } p = 11, \\ \{25, 40, 50\} & \text{if } p = 5, \\ \{27, 33, 48, 54, 66\} & \text{if } p = 3, \\ \{34, 40, 44, 48, 50, 54, 66\} & \text{if } p = 2, \\ \emptyset & \text{otherwise.} \end{cases}$$
  
 • In particular,  $\exists$  RDP K3 with a  $\mu_p$ -action in char  $p \iff p \leq 19$ .

### Proof of A (symplectic case)

- Suffices to consider (symplectic)  $\mu_p$ -actions.  
 Write  $\pi: X \rightarrow Y = X/\mu_p$ . Then  $\mathcal{O}_Y = (\mathcal{O}_X)_0$ .  
**Lemma.** Suppose  $z \in X$ : a fixed point (smooth or RDP), and  $\mu_p$ -action symplectic at  $z$ . Then  $z$  is an isolated fixed point, and  $\pi(z)$  is an RDP.  
*Proof.* If  $z$  is smooth, then  $\exists x, y$ : coordinate with weight  $a, b \in \mathbb{Z}/p\mathbb{Z}$ ,  $\neq 0$ . We have  $a + b = 0$  since symplectic. Then  $\hat{\mathcal{O}}_{Y, \pi(z)} = k[[x^p, xy, y^p]]$ : RDP of type  $A_{p-1}$ .  
 If  $z$  is an RDP, consider  $\mu_p \curvearrowright \text{Bl}_z X$  and reduce to the smooth case.  
**Cf:** action of finite subgroup of  $\text{SL}_2(\mathbb{C})$ .  
**Lemma.** Suppose  $z \in X$ : a non-fixed RDP. Then  $\pi(z)$  is smooth or RDP.  
**Remark.** This is a new feature.  
**Example.**  $A_{pm-1}$  ( $xy + z^{pm} = 0$ ), with action  $\text{wt}(x, y, 1 + z) = (0, 0, 1)$ : quotient is  $A_{m-1}$  ( $xy + Z^m = 0$  where  $Z = z^p$ ).  
**Lemma.** Outside the fixed locus and the RDPs,  $(\pi_*(\Omega_X^2))_0 \cong \Omega_Y^2$ .  
 From the lemmas,  $Y$  has only RDPs as singularities, and  $H^0(Y^{\text{sm}}, \Omega_Y^2)$  is 1-dim.

### Proof of C

- If  $p \nmid n$ , this follows from Keum's result. We consider the case  $p \mid n$ . Write  $n = p^e r$ ,  $p \nmid r$ .  
 $X$  has an RDP  $z$  (since a smooth K3 has no  $\mu_p$ -action). We may assume it is not fixed by  $\mu_{p^e}$  (otherwise, blow it up).  
 We classify all non-fixed actions on RDPs. and check that  $n \in S_\mu(0) \setminus E'_p$  in all cases.  
 Typical case:  $p \neq 2$ , the  $\mu_r$ -orbit of  $z$  has  $r/2$  elements, each of type  $A_{p^e-1}$ , and  $\mu_2$  acts on  $z$  non-symplectically. In this case we deduce  $(r/2)(p^e - 1) < b_2(K3) = 22$ . This almost implies  $n \in S_\mu(0) \setminus E'_p$ , and we kill few exceptions individually.

### Examples for C: $n = 34$

- $X(p): y^2 = x^3 + t^7 x + t^2$  in char  $p$ .  
 •  $X(0)$  is Kondo's example of an RDP K3 ( $A_1, A_2$ ) with a  $\mu_{34}$ -action ( $\text{wt}(x, y, t) = (4, 23, 6)$ ) or equivalently a  $\mathbb{Z}/34\mathbb{Z}$ -action.  
 •  $X(17)$  is an RDP K3 ( $A_1, A_2, A_{16}$ ) with a  $\mu_{34}$ -action (but not with a  $\mathbb{Z}/34\mathbb{Z}$ -action).  
 •  $X(2)$  is not an RDP K3 (a non-RDP singularity the at origin). The true char 2 reduction of  $X(0)$  is  $y'(y' - t') = x'^3 + t'^7 x'$ , and does not admit a  $\mu_{34}$ -action (but instead admits a  $\mathbb{Z}/34\mathbb{Z}$ -action).